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SOME FIXED POINT THEOREMS FOR COMPACT MAPS AND FLOWS IN BANACH SPACES(1)

BY
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Abstract. Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X, with S_0 and S_2 closed and S_1 open in S_2. If f is a compact mapping of S_2 into X such that $\bigcup_{j=1}^{m} f^j(S_1) \subset S_2$ and $f^m(S_1) \cup f^{m+1}(S_1) \subset S_0$ for some $m>0$, then f has a fixed point in S_0. (This extends a result of F. E. Browder published in 1959.) Also, if $\{T_t : t \in R^+\}$ is a continuous flow on the Banach space X, $S_0 \subset S_1 \subset S_2$ are convex subsets of X with S_0 and S_2 compact and S_1 open in S_2, and $T_{t_0}(S_1) \subset S_0$ for some $t_0>0$, where $T_t(S_1) \subset S_2$ for all $t \leqq t_0$, then there exists $x_0 \in S_0$ such that $T_t(x_0)=x_0$ for all $t \geqq 0$. Minor extensions of Browder's work on "nonejective" and "nonrepulsive" fixed points are also given, with similar results for flows.

Interest in fixed point theorems in Banach and locally convex topological linear spaces has been increasing during the past few years as more applications are found in the field of differential and integral equations ([4], [7], [9], [10], [11], and [12]). An important class of fixed point theorems deals with compact or completely continuous mappings on convex sets, going back to the Schauder [13] and Tychonoff [14] theorems, wherein a convex set is assumed mapped into itself, and continuing to more complicated recent theorems such as those of F. E. Browder ([1], [2], and [3]) and others.

The theorems of [1], [2], and [3] deal with the action of iterates of a given mapping, rather than the mapping itself, and it is the purpose of this paper to extend these theorems in the same vein and to prove analogous theorems related to flows, i.e., semigroups of mappings indexed by the nonnegative real half line. This latter extension was suggested to the author by A. J. Goldman of the National Bureau of Standards.

A. **Extensions of Browder's first result.** This section will generalize the work of Browder in [1]. The result to be generalized is Theorem 2 of [1].

Let S and S_1 be open convex subsets of the Banach space X, S_0 a closed convex subset of X, $S_0 \subset S_1 \subset S$, f a compact mapping of S into X. Suppose that, for a

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positive integer m, f^m is well defined on S_1, $\bigcup_{j=0}^{m-1} f^j(S_0) \subset S_1$, while $f^m(S_1) \subset S_0$. Then f has a fixed point in S_0.

The first lemma is a restatement of Lemma 1 of [1].

LEMMA 1. *Let A be a simplicial complex and suppose that f is a self-mapping of A such that for some integer $m > 0$, $f^m(A)$ is contained in a closed acyclic subset B of A. Then f has a fixed point.*

Lemma 2 gives the basic innovation on the work of [1]. A variant of this lemma was originally published in [8] and another variant was given by Browder in [2].

LEMMA 2. *Given a complex K, let K_0 be a subcomplex and C_0 a closed, bounded, acyclic subset of K_0. Suppose that f is a simplicial mapping of the nth barycentric subdivision of K into K such that, for some positive integer m, $f^j(K_0) \subset C_0$ for $m \leqq j \leqq 2m-1$. Then f has a fixed point in C_0.*

Proof. Let $A = \bigcup_{j=0}^{m-1} f^j(K_0)$. Then A is a closed subcomplex of K and $f(A) \subset A$, since $f^m(K_0) \subset C_0 \subset A$. Furthermore, $f^m(A) \subset C_0$, a closed, bounded, acyclic subset of A. By Lemma 1, f has a fixed point in A, and, since $f^m(A) \subset C_0$, this fixed point must be in C_0.

LEMMA 3. *In a Banach space X, let f be a uniformly continuous mapping of a set K into itself. For any given integer $m > 0$ and any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that, if $g: D \to D$ is a self-mapping of the subset $D \subset K$ and $\|g(x) - f(x)\| < \delta$ on D, then $\|g^j(x) - f^j(x)\| < \varepsilon$ on D for $1 \leqq j \leqq m$.*

Proof. Obviously the statement is true for $m = 1$. Assume it true for $1, 2, \ldots, m-1$. Then

$$\|g^m(x) - f^m(x)\| \leqq \|g(g^{m-1}(x)) - f(g^{m-1}(x))\| + \|f(g^{m-1}(x)) - f(f^{m-1}(x))\|.$$

If we choose δ so that $\|g(x) - f(x)\| < \varepsilon/2$ on D, $\|g^j(x) - f^j(x)\| < \varepsilon$ for $2 \leqq j \leqq m-1$, and so that $\|g^{m-1}(x) - f^{m-1}(x)\| < \eta$, where $\|f(x) - f(y)\| < \varepsilon/2$ whenever $\|x - y\| < \eta$, then the result follows.

The fourth lemma is a restatement of Lemma II.7 of [6]. (This lemma may also be taken as a special case of Lemma 9, Dugundji's extension theorem.)

LEMMA 4 (GRANAS). *There exists a retraction of any Banach space onto any closed, convex, separable subset of itself. In particular, there exists a retraction of a Banach space onto any compact, convex subset of itself.*

LEMMA 5. *Let X be a finite-dimensional linear topological space and let $S_0 \subset S_1 \subset S_2$ be bounded convex sets of X such that S_0 and S_2 are closed and S_1 is a neighborhood of S_0, relative to S_2. Let $f: S_2 \to X$ be a continuous map such that for some integer $m > 0$ we have*

(1) $$f^j(S_1) \subset S_2, \qquad 1 \leqq j \leqq m-1,$$

and

$$(2) \qquad f^j(S_1) \subset S_0, \qquad m \leqq j \leqq 2m-1.$$

Then f has a fixed point in S_0.

Proof. We may assume that $f(S_2) \subset S_2$, since if this is not the case we can consider a new map $\bar{f} = rf(x)$ where r is a retraction of X onto S_2, which exists by Lemma 4. It is clear that (1) and (2) are satisfied for $\bar{f}$ and that any fixed point of $\bar{f}$ in S_0 is a fixed point of f.

Now let $T: K_\delta \to S_2$ be a triangulation of S_2 of mesh δ. Then for some barycentric subdivision $B_r(K_\delta)$, there exists a simplicial approximation $\bar{g}: B_r(K_\delta) \to K_\delta$ to the map f. Let $g: S_2 \to S_2$ be $\bar{g}$ transferred back to S_2. Then we have for $x \in S_2$,

$$\|f(x)-g(x)\| \leqq \|f(x)-f(v_i)\| + \|f(v_i)-g(v_i)\| + \|g(v_i)-g(x)\|,$$

where v_i is some vertex of a simplex of $B_r(K_\delta)$ (identified with S_2) containing x. Now $\|f(v_i)-g(v_i)\| < \delta$ and $g\|(v_i)-g(x)\| < \delta$, by the definition of simplicial approximation, and, since $\|x-v_i\| < \delta$, $\sup_{x \in S_2} \|f(x)-f(v_i)\| \to 0$ as $\delta \to 0$, by the uniform continuity of f. Thus

$$\eta = \sup_{x \in S_2} \|f(x)-g(x)\| \to 0 \quad \text{as } \delta \to 0.$$

Then by Lemma 3,

$$\eta_1 = \sup\{\|f^j(x)-g^j(x)\| : x \in S_2, j \in [1, 2m]\} \to 0 \quad \text{as } \delta \to 0.$$

Since S_1 is a relative neighborhood of S_0, we have for some $\varepsilon > 0$, that $N_\varepsilon(S_0) \cap S_2 \subset S_1$. Since $f^j(S_1) \subset S_0$ for $m \leqq j \leqq 2m-1$, by hypothesis (2), it is clear that

$$g^j(S_1) \subset \mathrm{Cl}\,(N_{\eta_1}(S_0)) \cap S_2$$

for $m \leqq j \leqq 2m-1$, by the definition of η_1 above. Let $S_1' = \mathrm{Cl}\,(N_{\varepsilon/2}(S_0)) \cap S_2$ and let K_1' be the subcomplex of K_δ consisting of those simplexes σ such that $\sigma \cap S_1' \neq \varnothing$. Then $S_1' \subset K_1'$, and it is also clear that if $\delta < \varepsilon/2$ then $K_1' \subset S_1$. Furthermore, if δ is also chosen so that $\eta_1 < \varepsilon/2$, then we have that

$$g^j(K_1') \subset g^j(S_1') \subset \mathrm{Cl}\,(N_{\eta_1}(S_0)) \cap S_2 \subset S_1'.$$

Applying Lemma 2 to the complex K_δ, the subcomplex K_1', and the acyclic subset (because of its convexity) S_1', we see that g has a fixed point in S_1', and in fact in $\mathrm{Cl}\,(N_{\eta_1}(S_0)) \cap S_2$.

Now let $\{\delta_n\}$ be a null sequence of δ's for which the above construction is possible, let $\{g_n\}$ be the corresponding g's, and let x_n be a fixed point of g_n, as found above. By the compactness of S_2, there exists a subsequence $x_{n(i)} \to x_0$. Then $f(x_0) = \lim g_{n(i)}(x_{n(i)}) = \lim x_{n(i)} = x_0$. Furthermore since $x_{n(i)} \in \mathrm{Cl}\,(N_{\eta_1}(S_0))$ and $\eta_1 \to 0$ as $\delta \to 0$, we must have $x_0 \in S_0$. This completes the proof.

THEOREM 6. *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X, with S_0 and S_2 compact and S_1 open relative to S_2. Let $f\colon S_2 \to X$ be a continuous mapping such that, for some integer $m > 0$,*

$$f^j(S_1) \subset S_2, \qquad 1 \leqq j \leqq m-1, \tag{1}$$

and

$$f^j(S_1) \subset S_0, \qquad m \leqq j \leqq 2m-1. \tag{2}$$

Then f has a fixed point in S_0.

Proof. We may assume that $f(S_2) \subset S_2$, since if this is not the case then by Lemma 4, there exists a retraction $r\colon X \to S_2$. We may then define a new map $\bar{f} = rf$ which has properties (1) and (2) and whose fixed points in S_0 are also fixed points of f.

Since S_1 is open in S_2 and S_0 is compact, there exists $\varepsilon > 0$ such that $N_\varepsilon(S_0) \cap S_2 \subset S_1$. Now by Lemma 3 there exists $\eta > 0$ such that for any map g defined on a subset D of S_2 we have $\|g^j(x) - f^j(x)\| < \varepsilon/2$ for $1 \leqq j \leqq 2m$ and for all $x \in D$ whenever $\|g(x) - f(x)\| < \eta$ for all $x \in D$.

Let $\{x_i\}$ be a finite collection of points in S_2, with some $x_j \in S_0$, such that for any $x \in S_2$ there exists an x_i with $\|x - x_i\| < \eta/5$. Let H be the finite-dimensional linear manifold generated by $\{x_i\}$. Let $R_0 = S_0 \cap H$, $R_1 = S_1 \cap H$, and $R_2 = S_2 \cap H$. Then R_0, R_1, and R_2 are nonempty convex sets in H. Therefore, there exists a triangulation $T\colon K \to R_2$ for some complex K. Since f is uniformly continuous on R_2, there exists $\delta > 0$ such that $\|f(x) - f(y)\| < \eta/5$ whenever $\|x - y\| < \delta$. We may assume that the mesh of the triangulation T is less than δ, since a barycentric subdivision will give this.

Define a map $g\colon R_2 \to R_2$ as follows. For each vertex $v \in R_2$ of the triangulation let $g(v)$ be some x_i such that $\|x_i - f(v)\| < \eta/5$. Extend g to all of R_2 by the rule: if $x \in R_2$ and $T^{-1}(x) = \sum \alpha_i T^{-1}(v_i)$, then $g(x) = \sum \alpha_i g(v_i)$.

Now if v_i and v_j are vertices of a common simplex in R_2, then we have

$$\begin{aligned} \|g(v_i) - g(v_j)\| &\leqq \|g(v_i) - f(v_i)\| + \|f(v_i) - f(v_j)\| + \|f(v_j) - g(v_j)\| \\ &< \eta/5 + \eta/5 + \eta/5 = 3\eta/5. \end{aligned}$$

Thus if x and y are any points of R_2 contained in the same simplex of K, then $\|g(x) - g(y)\| < 3\eta/5$ also, since the distance between $g(x)$ and $g(y)$ is no greater than the maximum distance between the g-images of any two vertices of the containing simplex. Therefore we have, for any $x \in R_2$,

$$\begin{aligned} \|f(x) - g(x)\| &\leqq \|f(x) - f(u)\| + \|f(u) - g(u)\| + \|g(u) - g(x)\| \\ &< \eta/5 + \eta/5 + 3\eta/5 = \eta, \end{aligned}$$

where u is any vertex of a simplex containing x.

From the above, plus the fact that $f^j(S_1) \subset S_0$ for $m \leqq j \leqq 2m-1$, we see that

$$g^j(R_1) \subset N_{\varepsilon/2}(f^j(R_1)) \subset N_{\varepsilon/2}(f^j(S_1)) \subset N_{\varepsilon/2}(S_0) \quad \text{for } m \leqq j \leqq 2m-1.$$

But $g\colon R_2 \to R_2$, and so we have

$$g^j(R_1) \subset N_{\varepsilon/2}(S_0) \cap R_2 = N_{\varepsilon/2}(R_0) \cap R_2 \quad \text{for } m \leqq j \leqq 2m-1.$$

Let $R_0' = N_{\varepsilon/2}(R_0) \cap R_2$. Then R_0' is closed and $N_{\varepsilon/2}(R_0') \cap H \subset R_1$. But $g^j(R_1) \subset R_0'$, for $m \leqq j \leqq 2m-1$, and so by the previous lemma g has a fixed point in R_0'.

Now let $\{\varepsilon_n\}$ be a null sequence of ε's as considered above, and let $\{g_n\}$ be the corresponding maps with fixed points $\{x_n\}$ respectively. By the compactness of S_2 there exists a subsequence $x_{n(i)} \to x_0$. By the reasoning of Lemma 5, $f(x_0) = x_0$ and $x_0 \in S_0$. This completes the proof.

THEOREM 7. *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X, with S_0 and S_2 closed and S_1 open relative to S_2. Let $f\colon S_2 \to X$ be a continuous, compact mapping such that for some integer $m > 0$,*

(1) $$f^j(S_1) \subset S_2, \qquad 1 \leqq j \leqq m-1,$$

and

(2) $$f^j(S_1) \subset S_0, \qquad m \leqq j \leqq 2m-1.$$

Then f has a fixed point in S_0.

Proof. Let $S_2' = h(f(S_2))$, where h is the convex closure operator. Let $R_0 = S_0 \cap S_2'$, $R_1 = S_1 \cap S_2'$, and $R_2 = S_2 \cap S_2'$. Then R_0 and R_2 are compact, and R_1 is open relative to R_2. Furthermore, we may define $\bar{f}\colon R_2 \to R_2$ by $\bar{f} = rf$, where r is any retraction of X onto R_2, which exists by Lemma 4. Then $\bar{f}^j(R_1) \subset R_2$ for $1 \leqq j \leqq m-1$ and $\bar{f}^j(R_1) \subset R_0$ for $m \leqq j \leqq 2m-1$, since $\bar{f}^j = f^j$ on R_1. Applying Theorem 6 to R_0, R_1, and R_2, we see that $\bar{f}$, and hence f, has a fixed point in $R_0 \subset S_0$.

COROLLARY 8. *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X, with S_0 and S_2 closed and S_1 open relative to S_2. Let $f\colon S_2 \to X$ be a continuous compact mapping such that, for some integer $m > 0$,*

(1) $$f^j(S_0) \subset S_1, \qquad 1 \leqq j \leqq m-1,$$

(2) $$f^j(S_1) \subset S_2, \qquad 1 \leqq j \leqq m-1,$$

and

(3) $$f^m(S_1) \subset S_0.$$

Then f has a fixed point in S_0.

Proof. By (1) and (3) we have that $f^j(S_1) \subset S_0$ for $2m \leqq j \leqq 4m-1$. Thus Theorem 7 applies.

Corollary 8 is very similar to Browder's theorem in [1].

The next lemma is a statement of an extension theorem proved by J. Dugundji in [5].

LEMMA 9 (DUGUNDJI). *Let X be an arbitrary metric space, A a closed subset of X, L a locally convex linear topological space, and f: $A \to L$ a continuous map. Then there exists an extension F: $X \to L$ of f. Furthermore, $F(X) \subset$ [convex hull of $f(A)$]*

The next theorem shows that the set S_2 of Theorems 6 and 7 is superfluous in some sense.

THEOREM 10. *Let $S_0 \subset S_1$ be convex subsets of the Banach space X with S_0 closed and S_1 open. Let f be a compact continuous mapping defined on some closed subset D of X such that $f^j(S_1) \subset D$ for all $j \geqq 0$ and such that, for some integer $m > 0$, $\bigcup_{j=m}^{2m-1} f^j(S_1) \subset S_0$. Then f has a fixed point in S_0.*

Proof. Note that the hypotheses imply $f^j(S_1) \subset S_0$ for $j \geqq m$. Let

$$A = \mathrm{Cl}\left(\bigcup_{j=0}^{m} f^j(S_1)\right).$$

Then $f(A) \subset A$. Furthermore, by Dugundji's theorem there exists an extension of the map $f|A$ to a map $\bar{f}$: $X \to X$. Since $\bar{f}|A = f|A$, we apply Theorem 7 to show that $\bar{f}$, and hence f, has a fixed point in S_0.

Next we show that the condition $\bigcup_{j=m}^{2m-1} f^j(S_1) \subset S_0$ is also unnecessarily strong, and, in fact, that only two iterates of f need take S_1 into S_0 to produce a fixed point for f (Corollary 13).

LEMMA 11. *Let N be any set of positive integers with c the greatest common factor of all $n \in N$. Then there exists a positive integer m and a subset $\{n_i\}_{i=1}^{k} \subset N$ such that all integers of the form cj, where $m \leqq j \leqq 2m$, can be written as*

$$cj = \sum_{i=1}^{k} a_i n_i$$

where the a_i's are nonnegative integers (dependent on j).

Proof. The lemma is well known and the proof may be found in a number of good books. See, for example, *Finite Markov Chains* by Kemeny and Snell, Van Nostrand, Princeton, N. J., 1960, pp. 6–7.

THEOREM 12. *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X with S_0 and S_2 closed and S_1 open relative to S_2. Let f: $S_2 \to X$ be a continuous, compact mapping such that, for some set of positive integers N,*

$$(1) \qquad f^j(S_1) \subset S_2, \qquad \text{for all } j \geqq 1,$$

and

$$(2) \qquad f^j(S_1) \subset S_0, \qquad j \in N.$$

Let $c = gcf(N)$. Then f^c has a fixed point in S_0.

Proof. By Lemma 11, there exists a subset $\{n_i\}_{i=1}^{k} \subset N$ such that for some positive integer m we may write $jc = \sum_{i=1}^{1} a_i n_i$ for each j such that $m \leqq j \leqq 2m$, where $a_i \geqq 0$ and the a_i depend on j. Thus

$$\begin{aligned} f^{jc}(S_1) &= f^{a_1 n_1}(f^{a_2 n_2}(\cdots(f^{a_k n_k}(S_1))\cdots)) \\ &\subset f^{a_1 n_1}(f^{a_2 n_2}(\cdots(f^{a_{k-1} n_{k-1}}(S_0))\cdots)) \\ &\subset \cdots \subset \cdots \subset S_0. \end{aligned}$$

Applying Theorem 7 completes the proof.

COROLLARY 13. *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X with S_0 and S_2 closed and S_1 open relative to S_2. Let $f\colon S_2 \to X$ be a continuous, compact mapping such that, for some integer $m > 0$,*

$$f^j(S_1) \subset S_2, \qquad 1 \leqq j \leqq m, \tag{1}$$

$$f^m(S_1) \cup f^{m+1}(S_1) \subset S_0. \tag{2}$$

Then f has a fixed point in S_0.

Proof. Since $f^j(S_1) \subset S_2$ for $1 \leqq j \leqq m$ and $f^m(S_1) \subset S_0 \subset S_1$, the iterates of f are well defined on S_1 for all $j \geqq 1$. Applying Theorem 12 gives us the desired result.

B. **Ejective and nonejective fixed points.** Closely related to the theorems of §A is the subject of a compact mapping of a convex set into itself, with fixed points whose neighborhoods eventually (in some sense) map outside of themselves. To be more precise, we introduce the terms "repulsive" and "ejective" fixed point, defined by Browder in [2] and [3]. Let C be a subset of the Banach space X, f a continuous mapping of C into itself. If x_0 is a fixed point of f, x_0 is said to be repulsive if there exists a neighborhood U of x_0 in C such that for every $x \in C - \{x_0\}$ there exists an integer $J(x)$ such that $f^j(x) \in C - U$ for $j \geqq J(x)$. Also, x_0 is said to be an ejective fixed point of f if there exists a (relative) neighborhood U of x_0 in C such that for any $x \in U - \{x_0\}$ there exists a positive integer $k(x)$ such that $f^{k(x)}(x) \in C - U$.

Now let f be a continuous mapping of the infinite-dimensional, compact, convex set C into itself. In [2] Browder proved that f has a nonrepulsive fixed point and in [3] the stronger result that f has a nonejective fixed point.

We would like to use the results of [3] to prove a theorem about mappings which take each point of the space, on some iterate, into a given bounded set. To start we have the following:

THEOREM 14. *Let C be an infinite-dimensional closed, convex set in the Banach space X and f a compact mapping of C into itself. Then f has a nonejective fixed point.*

Proof. Let $\{x_n\}$ be an infinite-dimensional sequence of points in C such that $d(x_n, f(C)) < 2^{-n}$. Clearly such a sequence exists. Let

$$C_1 = h(\bigcup \{x_n\} \cup f(C)) \subset C,$$

where h denotes the convex closure operator. Then $\bigcup \{x_n\} \cup f(C)$ is clearly compact, so that C_1 is also compact, and we have

$$f(C_1) \subset f(C) \subset C_1.$$

Thus, since C_1 is infinite-dimensional, by Browder's theorem f has a nonejective fixed point in C_1. But by the definition of ejectivity, a fixed point of f which is nonejective in C_1 is also nonejective in C. This proves the theorem.

THEOREM 15. *Let X be an infinite-dimensional Banach space and f a completely continuous mapping of X into itself such that for each $K>0$ there exists $k>0$ such that $\|f(x)\|>K$ whenever $\|x\|>k$. Suppose also that there exists a bounded set E such that for each $x \in X$ there is a positive integer $m=m(x)$ with $f^m(x) \in E$. Then f has a nonejective fixed point in E.*

Proof. First we show that $X-f(X)$ contains an open set. Let B_a denote the open ball about the origin of radius a. Given $K>0$, we can find $k>0$ such that

$$f(X-\mathrm{Cl}\,(B_k)) \subset X-\mathrm{Cl}\,(B_K).$$

Furthermore, $f(\mathrm{Cl}\,(B_k))$ is a compact set, since f is completely continuous, and so $B_K-f(\mathrm{Cl}\,(B_k))$ is a nonvoid open set. Thus, since $B_K-f(\mathrm{Cl}\,(B_k)) \subset X-f(X)$, there is an open set contained in $X-f(X)$.

Let x_0 be an interior point of $X-f(X)$, and we may assume without loss of generality that $x_0=0$. Then there exists ε such that $N_\varepsilon(0)=B_\varepsilon \subset X-f(X)$. Thus f maps $X-B_\varepsilon$ into itself. Also, there exists B_r containing B_ε and E, so that for any $x \in X$ there is a positive integer $m=m(x)$ such that $f^m(x) \in B_r$. Now let the involution $\varphi: X-\{0\} \to X-\{0\}$ be defined by $\varphi(x)=x/\|x\|^2$. Let $g: X-\{0\} \to X-\{0\}$ be defined by $g=\varphi f \varphi$. Extend g to all of X by defining $g(0)=0$. Now g is continuous on $X-\{0\}$, since f is continuous, and g is continuous at 0, since for a given $\eta>0$, there exists $\delta>0$ such that $\|f(x)\|>1/\eta$ whenever $\|x\| \geqq 1/\delta$. Thus $\|g(x)\|<\eta$ whenever $\|x\|<\delta$.

Also, since $f(X-B_\varepsilon) \subset X-B_\varepsilon$, we have $g(\mathrm{Cl}\,(B_{1/\varepsilon})) \subset \mathrm{Cl}\,(B_{1/\varepsilon})$.

Next we show that $g|\mathrm{Cl}\,(B_{1/\varepsilon})$ is compact. For if $\{x_n\}$ is any sequence of points in $\mathrm{Cl}\,(B_{1/\varepsilon})$, then if $\{x_n\}$ has 0 as a limit point so does $\{g(x_n)\}$.

If $\{x_n\}$ does not have 0 as a limit point, then the x_n's are bounded away from 0, so that $\{\varphi(x_n)\}$ is a bounded sequence. Thus $\{f\varphi(x_n)\}$ has a limit point (in $X-B_\varepsilon$), by the complete continuity of f, so that $\{g(x_n)\}$ also has a limit point. This shows that every infinite set of points in $g(\mathrm{Cl}\,(B_{1/\varepsilon}))$ has a limit point, and so $g|\mathrm{Cl}\,(B_{1/\varepsilon})$ is a compact mapping.

Finally, 0 is an ejective fixed point of $g|\mathrm{Cl}\,(B_{1/\varepsilon})$. For if $x \in B_{1/r}$, $x \neq 0$, then for some $m>0$ we have $f^m\varphi(x) \in E \subset B_r$. Thus $g^m(x)=\varphi f^m \varphi(x) \in X-B_{1/r}$.

Applying Theorem 14, we see that $g|B_{1/\varepsilon}$ has a nonejective fixed point x_0 in $\mathrm{Cl}\,(B_{1/\varepsilon})$. Now $x_0 \neq 0$, since 0 is ejective, and if x_0 is nonejective under g then $\varphi(x_0)$ is nonejective under f. Thus $\varphi(x_0)$ is a nonejective fixed point of f which must lie in E, since $f^m\varphi(x_0) \in E$ for some $m>0$. This completes the proof.

If considerations of ejectivity of the fixed point can be neglected then a much stronger result can be obtained, as shown by the following theorem.

THEOREM 16. *Let f be a completely continuous mapping of the Banach space X into itself. Suppose that there exists a bounded set E such that for each $x \in X$ there exists $m = m(x)$ such that $f^m(x) \in E$. Then f has a fixed point in E.*

Proof. Let T be any closed ball containing E in its interior and let $R = \mathrm{Cl}\,(f(T))$, a compact set. From the hypotheses, for each $x \in R$ there exists $m = m(x)$ such that $f^m(x) \in E$. But since $E \subset T^0$ there exists $\varepsilon = \varepsilon(x)$ such that $f^m(N_\varepsilon(x)) \subset T$. Assign such an $m(x)$ and $\varepsilon(x)$ to each $x \in R$. Then the collection of open sets $\{N_{\varepsilon(x)}(x)\}$ covers R, and by the compactness of R there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ of points in R, with associated sets $\{m_1, m_2, \ldots, m_n\}$ and $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, such that

$$f^{m_i}(N_{\varepsilon_i}(x_i) \cap R) \subset T \quad \text{and} \quad R \subset \bigcup_{i=1}^{n} N_{\varepsilon_i}(x_i).$$

Let $m_0 = \max\{m_i\}$ and let $S_0 = h(\bigcup_{j=0}^{m_0+1} f^j(T))$, where h is the convex closure operator. Let $S_1 = N_\eta(S_0)$, where η is any positive number.

For any $x \in T$ it is easy to show that $f^j(x) \in S_0$ for any $j \geqq 0$. For if $x \in T$, assume $f^j(x) \in S_0$ for $j \leqq j_0$. If $j_0 \leqq m_0$ then it is obvious that $f^{j_0+1}(x) \in S_0$, by the definition of S_0. If $j_0 > m_0$, then $f(x) \in R$ and hence $f^{k+1}(x) \in T$ for some $k \leqq m_0$. Thus

$$f^{j_0}(x) = f^{j_0-k-1}(f^{k+1}(x)) \in f^{j_0-k-1}(T),$$

and the result holds by induction.

Now it may be shown in the same way as above that there exists some positive integer m' such that, for each $x \in S_1$, $f^j(x) \in T$ for some $j \leqq m'$. But then $f^j(x) \in S_0$ for all $j \geqq m'$ by the above. Letting $S_2 = h(\bigcup_{j=0}^{m'} f^j(S_1))$ and applying Theorem 7 to S_0, S_1, and S_2, we see that f has a fixed point in S_0. But for any $x \in S_0$, $f^j(x) \in E$ for some $j > 0$. Thus this fixed point must lie in E, thereby proving the theorem.

C. **Extensions to flows.** Many of the previous theorems have analogues in the theory of flows. In this case, proof of the existence of a common fixed point of all the members of the flow, a so-called stationary point, is the desired result to be obtained.

Let $\{T_s : s \in S\}$ be a set of continuous mappings of a subset Y of the Banach space X into itself, where S is a commutative topological semigroup with identity element 0 such that

$$T_0(x) = x, \qquad x \in X,$$

and

$$\begin{aligned} T_s(T_t(x)) &= T_t(T_s(x)) \\ &= T_{t+s}(x), \qquad x \in X; \quad s, t \in S, \end{aligned}$$

and satisfying the continuity condition that for each $t \in S$

$$\sup\{\|T_t(x) - T_s(x)\| : x \in Y\} \to 0 \quad \text{as } s \to t.$$

Then $\{T_s : s \in S\}$ is called an S-semigroup of operators on Y. We shall consider only the case where $S=R^+$, the nonnegative real numbers with the usual topology. In this case, $\{T_s : s \in R^+\}$ is called a flow.

The following theorems extend the previous results to flows([2]).

THEOREM 17. *Let $\{T_s : s \in R^+\}$ be a flow on the subset Y of the Banach space X. Let $C_0 \subset C_1 \subset C_2$ be convex subsets of Y such that C_0 and C_2 are compact and C_1 is a neighborhood of C_0 relative to C_2. Suppose further that for some closed interval $[a, b]$ with $0 \leqq a < b$ we have*

$$T_s(C_1) \subset C_2, \qquad s \in [0, a],$$
$$T_s(C_1) \subset C_0, \qquad s \in [a, b].$$

Then there exists a point $x_0 \in C_0$ such that $T_s(x_0)=x_0$ for all $s \in R^+$.

Proof. For $K=2/(b-a)$, we have that for any integer $k>K$ there exists an integer m such that $a<m/k<(m+1)/k<b$. Thus $T_{m/k}(C_1)\subset C_0$ and $T_{(m+1)/k}(C_1)\subset C_0$. That is, $T^m_{1/k}(C_1)\subset C_0$ and $T^{m+1}_{1/k}(C_1)\subset C_0$. Thus by Corollary 13 $T_{1/k}$ has a fixed point $x_k \in C_0$. Since C_0 is compact, there exists a limit point x_0 of the set $\{x_n\}$ and a subsequence of $\{x_n\}$, which we denote by $\{y_n\}$, such that $y_n \to x_0$.

Now let T_s be any member of the flow. For any interger $k>0$, there exists an integer $j \geqq 0$ such that $|s-j/k|<1/k$. Let $g_k=T_{j/k}$. Then $g_k(x_k)=x_k$ and $g_k \to T_s$. For each $y_n=x_{m_n}$ of the previously determined subsequence, let $f_n=g_{m_n}$. Thus $y_n \to x_0$, $f_n(y_n)=y_n$, and $f_n \to T_s$. From this we have

$$T_s(x_0) = \lim_{n\to\infty} f_n(y_n) = \lim_{n\to\infty} y_n = x_0.$$

This completes the proof.

Even this result is not the strongest possible. We may also have $b=a$, that is, $T_a(C_1)\subset C_0$ only, and a common fixed point will still exist.

THEOREM 18. *Let $\{T_s : s \in R^+\}$ be a flow on the subset Y of the Banach space X. Let $C_0 \subset C_1 \subset C_2$ be convex subsets of Y such that C_0 and C_2 are compact and C_1 is a neighborhood of C_0 relative to C_2. Suppose further that for some $a>0$ we have*

$$T_s(C_1) \subset C_2, \qquad s \in [0, a],$$
$$T_a(C_1) \subset C_0.$$

Then there exists a point $x_0 \in C_0$ such that $T_s(x_0)=x_0$ for all $s \in R^+$.

Proof. Since C_0 is compact, there exists $\varepsilon>0$ such that $N_\varepsilon(C_0) \cap C_2 \subset C_1$. For any $\eta<\varepsilon$, $\eta>0$, there exists $\delta>0$ such that $\|T_s(x)-T_a(x)\|<\eta$ whenever $|s-a|\leqq\delta$, for all $x \in Y$, by the continuity of the flow. Thus

$$T_s(C_1) \subset \mathrm{Cl}\,(N_\eta(C_0) \cap C_2) = C_0'$$

([2]) It has been brought to the author's attention by G. S. Jones that several of the results on flows presented here could also be obtained from the Schauder theorem, using a theorem on continuous continuation of mappings (cf. [6, Theorem V.2, p. 49]).

for $s \in [a-\delta, a]$ and so there is a stationary point z of the flow in C_0', by the previous theorem. Let $\{\eta_n\}$ be a null sequence, $C_0^{(n)} = N_{\eta_n}(C_0) \cap C_2$, and z_n be the stationary point of the flow in $C_0^{(n)}$ found by the above. It is clear that the sequence $\{z_n\}$ has a limit point in C_0 which must also be a stationary point of the flow. This completes the proof.

An extension to flows is also possible for Browder's theorem on nonejective fixed points. If $\{T_s : s \in R^+\}$ is a flow on the set C, we define x_0 to be an ejective stationary point of the flow if $T_s(x_0) = x_0$ for all $s \in R^+$ and for some neighborhood U of x_0 in C we have that for each $x \in U - \{x_0\}$ there exists $s = s(x) > 0$ such that $T_s(x) \in C - U$.

THEOREM 19. *Let $\{T_s : s \in R^+\}$ be a flow on the infinite-dimensional compact, convex set C in the Banach space X. Then $\{T_s\}$ has a nonejective stationary point.*

Proof. From Browder's theorem, each T_s has a nonejective point in C. Let $\{T_{s_n}\}$ be a sequence of maps with $s_n > 0$, $s_n \to 0$, and let x_n be a nonejective fixed point of T_{s_n}. We may assume, without loss of generality, that $x_n \to x_0$. As shown in the proof of Theorem 17, x_0 is then a fixed point for all T_s.

Now suppose that x_0 is an ejective stationary point. By the definition of ejectivity, this means that there is a neighborhood U of x_0 in C such that for any neighborhood $U_1 \subset U$ we have that for each $x \in U_1 - \{x_0\}$ there exists $t = t(x) > 0$ such that $T_t(x) \in C - U_1$. Let $\varepsilon > 0$ be such that $V = N_\varepsilon(x_0) \cap C \subset U$ and let $V_1 = N_{\varepsilon/2}(x_0) \cap C$. Then for each $x \in V_1 - \{x_0\}$ there exists $t = t(x) > 0$ such that $T_t(x) \in C - V$.

Consider any point of the previous sequence $x_n \in V_1$. Since x_n is a nonejective fixed point of T_{s_n}, there exists some $z_n \in V_1$ such that $z_n \neq x_0$ and $T_{s_n}^k(z_n) \in V_1$ for all integers $k \geqq 0$. But by the ejectivity of x_0, there exists $t_n > 0$ such that $T_{t_n}(z_n) \in C - V$. Let $k \geqq 0$ be an integer such that

$$|ks_n - t_n| < s_n, \quad \text{and} \quad ks_n \leqq t_n.$$

Then

$$T_{t_n}(z_n) = T_{t_n - ks_n}(T_{ks_n}(z_n)) = T_{t_n - ks_n}(y_n)$$

where $y_n \in V_1$ and $T_{t_n - ks_n}(y_n) \in C - V$. Thus

$$\|y_n - T_{t_n - ks_n}(y_n)\| \geqq \varepsilon/2.$$

But by the continuity of the flow at T_0, there exists $\delta > 0$ such that

$$\|T_t(x) - x\| < \varepsilon/2$$

for all $t < \delta$, and all $x \in C$. Since $\lim_{n\to\infty} t_n - ks_n = 0$, we have established a contradiction and the theorem is proved.

The final theorem on flows is an extension of Theorem 16. We shall consider only the finite-dimensional case, since it would seem to be difficult, if not impossible, to find a flow $\{T_s\}$ on an infinite-dimensional space such that T_s is completely continuous for $s > 0$ and T_0 is the identity.

THEOREM 20. *Let X be a finite-dimensional Banach space and $\{T_s : s \in R^+\}$ a flow on X. Suppose there exists a bounded set E such that for each $x \in X$ there exists $s = s(x)$ such that $T_s(x) \in E$. Then there is a stationary point of the flow in E.*

Proof. Let B_r be a closed ball about the origin of radius r containing E. By the continuity of the flow at T_0, there exists $\delta > 0$ such that $\|T_t(x) - x\| < r$ for $t < \delta$. Let $\{t_n\}$ be a null sequence with each $t_n < \delta$. For each $x \in X$, there exists $s = s(x)$ such that $T_s(x) \in E \subset B_r$. Let $k_n = k_n(x)$ be a nonnegative integer such that

$$|k_n t_n - s| < t_n < \delta \quad \text{and} \quad k_n t_n < s.$$

Since $T_s(x) \in B_r$ and

$$\|T_s(x) - T_{k_n t_n}(x)\| = \|T_{s - k_n t_n}(T_{k_n t_n}(x)) - T_{k_n t_n}(x)\| < r,$$

we have that $T_{k_n t_n}(x) = T_{t_n}^{k_n}(x) \in B_{2r}$. Since such a k_n exists for each x, we may apply Theorem 16 to show that T_{t_n} has a fixed point $x_n \in E$. As shown in Theorem 17, a limit point x_0 of the sequence $\{x_n\}$ is a stationary point of the flow. Since $T_{s_0}(x_0) \in E$ for some $s_0 > 0$, we must have $x_0 \in E$, completing the proof.

D. **Conjectures and examples.** One feels that the "intermediate" set S_1 in Theorem 12 is not really necessary, so that the following might be conjectured:

Let $S_0 \subset S_2$ be compact, convex subsets of the Banach space X and f a continuous map of S_2 into X such that, for some set of positive integers N,

(1) $$f^j(S_0) \subset S_2 \quad \text{for all } j \geqq 1$$

and

(2) $$f^j(S_0) \subset S_0, \qquad j \in N.$$

Let $c = gcf(N)$. Then f^c has a fixed point in S_0.

This could be proved, for instance, if there exists a sequence of maps $\{f_n\}$, where $f_n \colon S_2 \to X$, $f_n \to f$, and a sequence of intermediate convex sets $\{S_1^n\}$ such that $S_0 \subset S_1^n$, S_1^n is open in S_2, $f_n^j(S_1^n) \subset S_2$ for all $j \geqq 1$, $f_n^j(S_1^n) \subset S_0$ for $j \in N$, and $S_1^n \to S_0$. For then each f_n^c has a fixed point in S_1^n and a limit point of these points must lie in S_0 and be fixed under f^c.

Also, since we may show as in the proof of Theorem 12 that for some $m > 0$ we have $f^{jc}(S_0) \subset S_0$ for $j \geqq m$, if there exists a retraction r taking $[N_\varepsilon(S_0) \cap S_2] \cup \bigcup_{j=1}^m f^{jc}(S_0)$ onto $\bigcup_{j=0} f^{jc}(S_0)$, then it is possible to define $\bar{f} = f^c r$ on $[N_\varepsilon(S_0) \cap S_2] \cup \bigcup_{j=1}^m f^{jc}(S_0)$. By Dugundji's theorem, $\bar{f}$ may be extended to a map g taking all of S_2 into itself. Letting $S_1 = N_\varepsilon(S_0) \cap S_2$, we see that $g^j(S_1) \subset S_0$ for all $j \geqq m$, and so Theorem 6 applies.

A second conjecture which, if true, would prove the first is the following:

Let S be a compact, convex subset of the Banach space X, and $f, g \colon S \to S$ be continuous, commuting maps. Then there exists a point $x_0 \in S$ such that $f(x_0) = g(x_0)$.

For if f satisfies the hypotheses of the first conjecture, then again there exists $m>0$ such that $f^{jc}(S_0)\subset S_0$ for $j\geqq m$. Thus $f^{mc}, f^{(m+1)c}: S_0\to S_0$ and so there exists x_0 such that

$$f^{mc}(x_0) = f^{(m+1)c}(x_0) = f^c(f^{mc}(x_0));$$

thus $f^{mc}(x_0)$ is a fixed point of f^c.

The second conjecture is true if either f or g is a homeomorphism onto, for then we have that either $f^{-1}g$ or $g^{-1}f$ takes S into itself and has a fixed point x_0, so that $f(x_0)=g(x_0)$. In fact, it is not even necessary that f and g commute.

The conjecture is also true if f, for example, is 1-1 and $g(S)\subset f(S)$. For then $f^{-1}g$ takes S into itself, and since S is compact f^{-1} is continuous, and hence $f^{-1}g$ is continuous. Thus $f^{-1}g$ has a fixed point x_0, and so $g(x_0)=f(x_0)$.

Finally, the conjecture is true if S is one-dimensional, as we now prove.

THEOREM 21. *Let I be the unit interval and f, g: $I\to I$ be continuous, commuting maps. Then there exists $x_0\in I$ such that $f(x_0)=g(x_0)$.*

Proof. First note that if f (say) is onto then the theorem is true. For there exist a, $b\in I$ such that $f(a)=0$, $f(b)=1$. Thus $(f-g)(a)\leqq 0$ and $(f-g)(b)\geqq 0$, implying $f=g$ at some x between a and b.

Now let f and g be any continuous commuting functions. Let $I_n=f^n(I)$. Then each I_n is a compact interval and $I_{n+1}\subset I_n$. Let $I_0=\bigcap_{n=1}^{\infty} I_n\neq\varnothing$. Then I_0 is also a compact interval and we have, for $x\in I_0$, $x\in I_n$ and hence there exists $y_n\in I$ such that $x=f^n(y_n)$. Thus

$$g(x) = gf^n(y_n) = f^n(g(y_n))\in I_n$$

and

$$f(x) = f^{n+1}(y_n)\in I_{n+1}.$$

It is clear then that f and g take I_0 into itself. If we can show that f is onto on I_0, then the proof will be complete.

Let $x\in I_0$. Then $x\in I_n$ so that, as before, there is $y_n\in I$ such that $x=f^n(y_n)$. Thus

$$x = f(f^{n-1}(y_n)) = f(z_n)$$

where $z_n=f^{n-1}(y_n)\in I_{n-1}$. Let z_0 be a limit point of the sequence $\{z_n\}$. Then it is clear that $z_0\in I_0$, and since $f(z_n)=x$ for all z_n we have $f(z_0)=x_0$. Thus $x\in f(I_0)$, completing the proof.

We now give an example to show the limitations on some of the theorems proved in the paper. Let T_θ be defined on the plane, using polar coordinates, by $T_\theta(\rho,\varphi)=(\rho,\varphi+\theta)$. Then $\{T_\theta\}$ is a flow on any disk S_2 about the origin. If S is any compact, convex set not containing 0 then $T_{2\pi}(S)=S$. However, T_θ has no fixed point in S for $\theta\neq 2m\pi$. This shows that Theorem 18 cannot be extended to the case where $C_0=C_1$.

Let $f_1 = T_{2\pi/m}$ in the above. Then for any compact, convex set S not containing 0 we have $f_1^m(S) \subset S$ and, if S is small enough, $f_1(S) \cap S = \varnothing$. It is clear that we can modify f_1 to a map f such that $f^m(N_\varepsilon(S)) \subset S$ for some $\varepsilon > 0$. But f^j has no fixed point in S for $j < m$. Thus it is seen that the number $c = gcf(N)$ in Theorem 12 is the best possible.

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